1 Babylonian mathematics

1 On beginnings

Obviously the pioneers and masters of hydraulic society were singularly well equipped to lay the foundations for two major and interrelated sciences: astronomy and mathematics. (Wittfogel, *Oriental Despotism*, p. 29, cited Høyrup 1994, p. 47)

Based on intensive cereal agriculture and large-scale breeding of small livestock, all in the hands of a centralized power, [this civilization] was quickly caught up in a widespread economy which made necessary the meticulous control of infinite movements, infinitely complicated, of the goods produced and circulated. It was to accomplish this task that writing developed; indeed for several centuries, this was virtually its only use. (M. Bottéro, cited in Goody 1986, p. 49)

When did mathematics begin? Naive questions like this have their place in history; the answer is usually a counter-question, in this case, what do you mean by 'mathematics'? A now rather outdated view restricts it to the logical-deductive tradition inherited from the Greeks, whose beginnings are discussed in the next chapter. The problem then is that much interesting work which we would commonly call 'mathematics' is excluded, from the Leibnizian calculus (strong on calculation but short on proofs) to the kind of exploratory work with computers and fractals which is now popular in studying complex systems and chaotic behaviour. Many cultures before and since the Greeks have used mathematical operations from simple counting and measuring onwards, and solved problems of differing degrees of difficulty; the question is how one draws the line to demarcate when mathematics proper started, or if indeed it is worth drawing.[1] As we shall see, the early history of Greek mathematics is hard to reconstruct with certainty. In contrast, the history of the much more ancient civilizations of Iraq (Sumer, Akkad, Babylon) in the years from 2500 to 1500 BCE provides a quite detailed, if still patchy record of different stages along a route which leads to mathematics of a kind. Without retracing the whole history in detail, in this chapter we can look at some of these stages as illustrations of the problem raised by our initial question/questions. Mathematics of what kind, and what for? And what are the conditions which seem to have favoured its development?

Before attempting to answer any of these questions, we need some minimal historical background. Various civilizations, with different names, followed each other in the region which is now Iraq, from about 4000 to 300 BCE (the approximate date of the Greek conquest). Our evidence about them is entirely archaeological—the artefacts and records which they left, and which have been excavated and studied by scholars. From a very early date, for whatever reason, they had, as the quotation from Bottéro describes, developed a high degree of hierarchy, slave or semi-slave labour, and obsessive bureaucracy, in the service of a combination of kings, gods, and

1. This relates to the questions raised recently in the field of 'ethnomathematics'; mathematical practices used, often without explicit description or justification, in a variety of societies for differing practical ends from divination to design. For these see, for example, Ascher (1991); because the subject is mainly concerned with contemporary societies, it will not be discussed in this book.

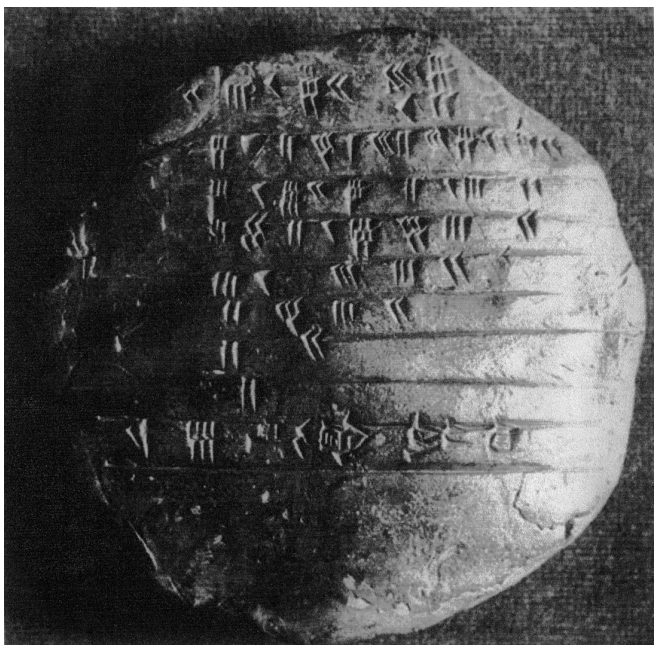
their priests. Writing of the most basic kind was developed around 3300 BCE, and continued using a more developed form of the original 'cuneiform' (wedge-shaped) script for 3000 years, in different languages. The documents have been unusually well preserved because the texts were produced by making impressions on clay tablets, which hardened quickly and were preserved even when thrown away or used as rubble to fill walls (see Fig. 1). A relatively short period in the long history has provided the main mathematical documents, as far as our present knowledge goes. As usual, we should be careful; our knowledge and estimation of the field has changed over the past 30 years and we have no way of knowing (a) what future excavation or decipherment will turn up and (b) what texts, currently ignored, will be found important by future researchers. In this period—from 2500 to 1750 BCE—the Sumerians, founders of a south Iraqi civilization based on Uruk, and inventors of writing among other things—were overthrown by a Semitic-speaking people, the Akkadians, who as invaders often do, adopted the Sumerian model of the state and used Sumerian (which is not related to any known language, and which gradually became extinct) as the language of culture. A rough guide will show the periods from which our main information on mathematics derives:

2500 BCE *'Fara period'.* The earliest (Sumerian) school texts, from Fara near Uruk; beginning of phonetic writing.

2340 BCE *'Akkadian dynasty'.* Unification of all Mesopotamia under Sargon (an Akkadian). Cuneiform is adapted to write in Akkadian; number system further developed.

2100 BCE *'Ur III'.* Re-establishment of Ur, an ancient Sumerian city, as capital. Population now mixed, with Akkadians in the majority. High point of bureaucracy under King Šulgi.

1800 BCE *'Old Babylonian', or OB.* Supremacy of the northern city of Babylon under (Akkadian) Hammurapi and his dynasty. The most sophisticated mathematical texts.



MS 1844

Fig. 1 A mathematical tablet (Powers of 70 multiplied by 2. Sumer, C. 2050 BC).

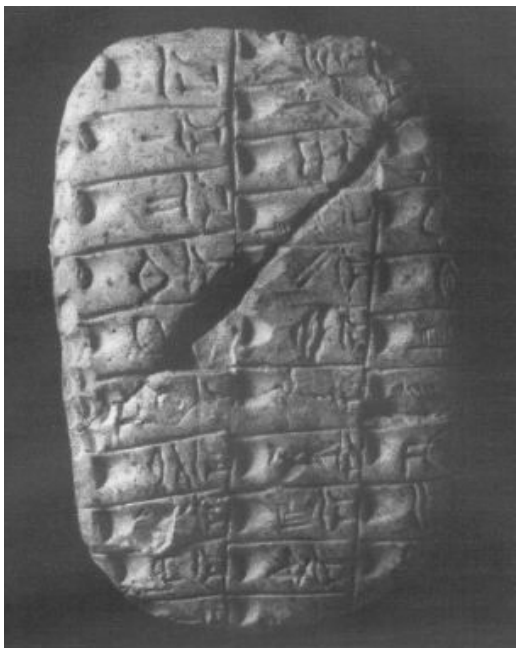


Fig. 2 Tablet VAT16773 (*c.* 2500 BCE).; numerical tally of different types of pigs.

Each dynasty lasted roughly a hundred years and was overthrown by outsiders, following a common pattern; so you should think of less-centralized intervals coming between the periods listed above. However, there was a basic continuity to life in southern Iraq, with agriculture and its bureaucratic-priestly control probably continuing without much change throughout the period.

In the quotation set at the beginning of the chapter, the renegade Marxist Karl Wittfogel advanced the thesis that mathematics was born out of the need of the ancient Oriental states of Egypt and Iraq to control their irrigation. In Wittfogel's version this 'hydraulic' project was indeed responsible for the whole of culture from the formation of the state to the invention of writing. The thesis has been attacked over a long period, and now does not stand much scrutiny in detail (see, for example, the critique by Høyrup 1994, p. 47); but a residue which bears examining (and which predates Wittfogel) is that the ancient states of Egypt and Iraq had a broadly similar priestly bureaucratic structure, and evolved both writing and mathematics very early to serve (among other things) bureaucratic ends. Indeed, as far as our evidence goes, 'mathematics' precedes writing, in that the earliest documents are inventories of goods. The development of counting-symbols seems to take place at a time when the things counted (e.g. different types of pigs in Fig. 2) are described by pictures rather than any phonetic system of writing. The bureaucracy needed accountancy before it needed literature—which is not necessarily a reason for mathematicians to feel superior.[2]

On this basis, there could be a case for considering the questions raised above with reference to ancient Egypt as well—the organization of Egyptian society and its use of basic mathematical procedures for social control were similar, if slightly later. However, the sources are much

2. There were certainly early poems celebrating heroic actions, the *Gilgamesh* being particularly famous. But in many societies, such poems are not committed to writing, and this seems to have been the case with the *Gilgamesh* for a long time—before it too was pressed into service by the bureaucracy to be learned by heart in schools.

poorer, largely because papyrus, the Egyptian writing-material, lasts so badly; there are two major mathematical papyri and a handful of minor ones from ancient Egypt. It is also traditional to consider Babylonian mathematics more 'serious' than Egyptian, in that its number-system was more sophisticated, and the problems solved more difficult. This controversy will be set aside in what follows; fortunately, the re-evaluations of the Babylonian work which we shall discuss below make it outdated. The Iraqi tradition is the earliest, it is increasingly well-known, discussed, and argued about; and on this basis we can (with some regret) restrict attention to it.

2 Sources and selections

> Even with great experience a text cannot be correctly copied without an understanding of its contents . . . It requires years of work before a small group of a few hundred tablets is adequately published. And no publication is 'final'. (Neugebauer 1952, p. 65)

> We need to establish the economic and technical basis which determined the development of Sumerian and Babylonian applied mathematics. This mathematics, as we can see today, was more one of 'book-keepers' and 'traders' than one of 'technicians' and 'engineers'. Above all, we need to research not simply the mathematical texts, but also the mathematical content of economic sources systematically. (Vaiman 1960, p. 2, cited Robson 1999, p. 3)

The quotations above illustrate how the study of ancient mathematics has developed. In the first place, crucially, there would not be such a study at all if a dedicated group of scholars, of whom Neugebauer was the best-known and most articulate, had not devoted themselves to discovering mathematical writings (generally in well-known collections but ignored by mainstream orientalists); to deciphering their peculiar language, their codes, and conventions; and to trying to form a coherent picture of the whole activity of mathematics as illustrated by their material—overwhelmingly, exercises and tables used by scribes in OB schools. These pioneers played a major role in undermining a central tenet of Eurocentrism, the belief that serious mathematics began with the Greeks. They pictured a relatively unified activity, practised over a short period, with some interesting often difficult problems. However, it is the fate of pioneers that the next generation discovers something which they had neglected; and Vaiman as a Soviet Marxist was in a particularly good position to realize that the neglected mathematics of book-keepers and traders was needed to complete the rather restricted picture derived from the scribal schools. For various reasons—its simplicity, based on a small body of evidence, and its supposed greater mathematical interest—the older (Neugebauer) picture is easy to explain and to teach; and you will find that most accounts of ancient Iraqi mathematics (and, for example, the extracts in Fauvel and Gray) concentrate on the work of the OB school tradition. In this chapter, trying to do justice to the older work and the new, we shall begin by presenting what is known of the classical (OB) period of mathematics; and then consider how the picture changes with the new information which we have on it and on its more practical predecessors.

At the outset—and this is implicit in what Neugebauer says—we have to face the problem of 'reading texts'. The ideal of a history in the critical liberal tradition, such as this aims to be, is that on any question the reader should be pointed towards the main primary sources; the main interpretations and their points of disagreement; and perhaps a personal evaluation. The reader is then encouraged to think about the questions raised, form an opinion, and justify it with reference to the source material. Was it possible to be an atheist in the sixteenth century; when was non-Euclidean geometry discovered, and by whom? There is plenty of material to support

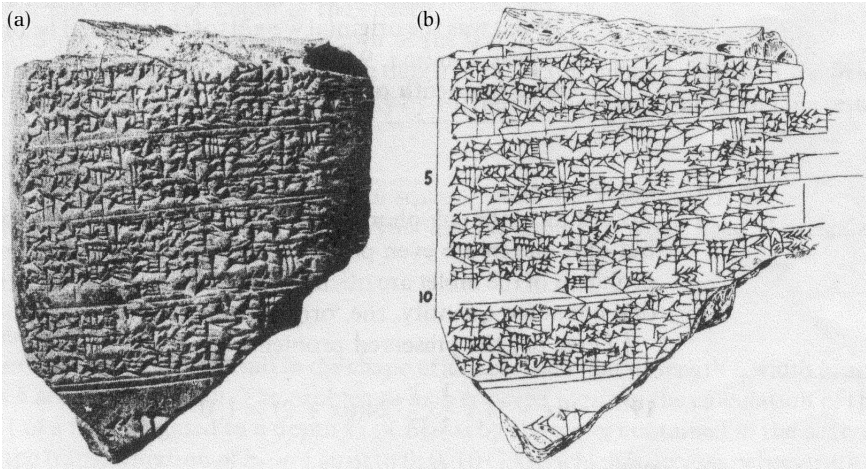


Fig. 3 The 'stone-weighing' tablet YBC4652; (a) photograph and (b) line drawing.

arguments on such questions, and there are writers who have used the material to develop a case. When we approach Babylonian mathematics, we find that this model does not work. There are, it is true, a large number of documents. They are partly preserved, sometimes reconstructed clay tablets, written in a dead language—Sumerian or Akkadian or a mixture—using the cuneiform script. It should also be noted that their survival is a matter of chance, and that we have few ways of knowing whether the selection which we have is representative. There seem to be gaps in the record, and most of our studies naturally are directed at the periods from which most evidence has survived.

Unless we want to spend years acquiring specialist knowledge, we must necessarily depend on experts to tell us how (a) to read the tablets, (b) to decipher the script, and (c) to translate the language.

It is useful to begin with an example. The tablet pictured (Fig. 3) is called YBC4652 (YBC for Yale Babylonian Catalogue). Here is the text of lines 4–6, which is cited in Fauvel and Gray as 1.E.1(20). The language is Akkadian, the date about 1800 BCE.

na_4 ì-pà ki-lá nu-na-tag 8-bi ì-lá 3 gín bí-daḫ-ma
igi-3-gál igi-13-gál a-rá 21 e-tab bi-daḫ-ma
ì-lá 1 ma-na sag na_4 en-nam sag na_4 $4\frac{1}{2}$ gín

Note that the figures in this quotation correspond to Babylonian numerals, of which more will follow later[3]; that is, where in the translation below the phrase 'one-thirteenth' appears, a more accurate translation would be '13-fraction', which shows that the *word* thirteen is not used. There is a special sign for $\frac{1}{2}$. The translation reads as follows (words in brackets have been supplied by the translator):

I found a stone, (but) did not weigh it; (after) I weighed (out) 8 times its weight, added 3 gín
one-third of one-thirteenth I multiplied by 21, added (it), and then
I weighed (it): 1 ma-na. What was the origin(al weight) of the stone? The origin(al weight) of the stone was $4\frac{1}{2}$ gín.

3. Except for the '4' in 'na_4', which seems to be a reference to the meaning of 'na' we are dealing with.

As you can see, from tablet to drawing to written Akkadian text to translation we have stages over which you and I have no control. We must make the best of it.

There are subsidiary problems; for example, we need to accept a dating on which there is general agreement, but whose basis is complicated. If a source gives the dates of King Ur-Nammu of the Third Dynasty as 'about 2111–2095 BCE', where do these figures come from, and what is the force of 'about'? Most scholars are ready to give details of all stages, but we are in no position to check. The restricted range of the earlier work perhaps made a consensus easier. In the last 30 years, divergent views have appeared. Even the traditional interpretation of the OB mathematical language has been questioned. An excellent account of this history is given by Høyrup (1996). In general the present-day historians of mathematics in ancient Iraq are models of what a secondary source should be for the student; they discuss their methods, argue, and reflect on them. But given the problems of script and language we have referred to, when experts do pronounce, by interpreting a document as a 'theoretical calculation of cattle yields', for example, rather than an actual count (see Nissen et al. 1993, pp. 97–102), the reader can hardly disagree, however odd the idea of doing such a calculation in ancient Ur may seem.

On a core of OB mathematics there is a consensus, which dates back to the pioneering work of Neugebauer and Thureau-Dangin in the first half of the twentieth century. There may be an argument about whether it is appropriate to use the word 'add' in a translation, but in the last instance there is agreement that things are being added. This is helpful, because it does give us a coherent and reliable picture of a practice of mathematics in a society about which a good deal is known. However, it is necessarily restricted in scope, and the sources which are usually available do not always make that fact clear. For example, most texts which you will see commented and explained come from the famous collection *Mathematical Cuneiform Texts* (Neugebauer and Sachs 1946). This is a selection, almost all from the OB period, and the selection was made according to a particular view of what was interesting. If you look at an account of Babylonian mathematics in almost any history book, what you see will have been filtered through the particular preoccupations of Neugebauer and his contemporaries, for whom OB mathematics was fascinating in part (as will be explained below) because it appeared both difficult and in some sense useless. The broader alternative views which have been mentioned do not often find their way into college histories.

It should be added that Neugebauer and Sachs's book is itself long out of print, and almost no library stocks it; your chances of seeing a copy are slim. Because the texts are so repetitive, the selections (from what is already a selection) given in textbooks, in particular Fauvel and Gray, give a pretty good picture of OB mathematics as it was known 50 years ago. All the same, they *are* selections from a large body of texts. Other useful reading—again not necessarily accessible in most libraries—is to be found in the works of Høyrup (1994), Nissen et al. (1993), and Robson (1999). There is a useful selection of Internet material (and general introduction) at http://it.stlawu.edu/~dmelvill/mesomath/; and in particular you can find various bibliographies, particularly the recent one by Robson (http://it.stlawu.edu/~dmelvill/mesomath/biblio/erbiblio.html).

Exercise 1. *(which we shall not answer). Consider the example given above; try to correlate the original text with (a) the pictures and (b) the translation. (Note that the line drawing is much clearer than the photograph; but, given that someone has made it, have we any reason to suspect its clarity?) Can you find out anything about either the script or the meaning of the words in the original as a result? How much editing seems to have been done, and how comprehensible is the end product?*

Exercise 2. *(which will be dealt with below). Clearly what we have here, in the translation, is a question and its answer. If I add the information that there are 60 gín in 1 ma-na, what do you think the question is, and how would you get at the answer?*

3 Discussion of the example

As is often observed, the problem above appears 'practical' (it is about weights of stones) until you look at it more closely. It was set, we are told, as an exercise in one of the schools of the Babylonian empire where the caste known as 'scribes' who formed the bureaucracy were trained in the skills they needed: literacy,[4] numeracy, and their application to administration. The usual answer to Exercise 2 is as follows. You have a stone of unknown weight (you did not weigh it); in our language, you would call the weight x gín. You then multiply the weight by 8 (how?) and add 3 gín, giving a weight of $8x + 3$. However, worse is yet to come. You now 'multiply one-third of one-thirteenth' by 21. What this means is that you take the fraction $\frac{1}{3} \times \frac{1}{13} \times 21 = \frac{21}{39}$ and multiply that by the $8x + 3$. You are not told that, but the tablets explain no more than they have to, and the problem does not come right without it, so we have to assume that the language which may seem ambiguous to us was not so to the scribes. Adding this, we have:

$$8x + 3 + \frac{21}{39}(8x + 3) = 60$$

Here we have turned the ma-na into 60 gín.

Clearly, as a way of weighing stones, this is preposterous; but perhaps it is not so very different from many equally artificial arithmetic problems which are set in schools, or were until recently. Effectively—and this is a point which we could deduce without much help from experts, although they concur in the view—such exercises were 'mental gymnastics' more than training for a future career in stone-weighing.

An advantage of beginning with the Babylonians is that their writing gives us a strong sense of historical *otherness*. Even if we can understand what the question is aiming at, the way in which it is put and the steps which are filled in or omitted give us the sense of a different culture, asking and answering questions in a different way, although the answer may be in some sense the same. In this respect, such writing differs from that of the Greeks, who we often feel are speaking a similar language even when they are not. You are asked a question; the type of question points you to a procedure, which you can locate in a 'procedure text'. To carry it out, you use calculations derived from 'table texts'; these tell you (to simplify) how to multiply numbers, to divide, and to square them. As James Ritter says:

> the systematization of both procedure and table texts served as a means to the same end: that of providing a network or grille through which the mathematical world could be seized and understood, at least in an operational sense. (Ritter 1995, p. 42)

It is worth noting that part of Ritter's aim in the text from which the above passage is taken is to situate the mathematical texts in relation to other forms of procedure, from medicine to divination, in OB society: they all provide the practitioner with 'recipes' of form: if you are confronted with

4. This included not only their own language but a dead language, Sumerian, which carried higher status; as civil servants in England 100 years ago had to learn Latin.

problem A, then do procedure B. The 'point' of the sum, then, is not mysterious, and indeed we can recognize in it some of our own school methods. First, scribes are trained to follow rules; second, they are required to use them to do something difficult. As usual, such an ability marks them off as workers by brain rather than by hand, and fixes their relatively privileged place in the social order. We know something of the arduous training and the beatings that went with it; but not what happened to those trainees who failed to make the grade.

What is mysterious in this particular case is the way in which one is supposed to get to the answer from the question, since the tablet gives no clue. Here the term 'procedure text' is rather a misnomer, but other tablets are more explicit on harder problems. With our knowledge of algebra, we can say (as you will find in the books) that the equation above leads to:

$$(8x + 3)\frac{39 + 21}{39} = 60$$

and so, $8x + 3 = 39$, and $x = 4\frac{1}{2}$. The fact that 39 and 21 add to 60, one would suppose, could not have escaped the setter of the problem; but language, such as I have just used would have been quite impossible. What method would have been available? The Egyptians (and their successors for millennia) solved simple linear equations, such as (as we would say) $4x + 3 = 87$ by 'false position': guessing a likely answer, finding it is wrong, and scaling to get the right one. This seems not to work easily in this case. To spend some time thinking about how the problem *could* have been solved is already an interesting introduction to the world of the OB mathematician.

Having looked at just one example, let us broaden out to the general field of OB mathematics. What were its methods and procedures, what was distinctive about it? And second, do the terms 'elementary' and 'advanced' make sense in the context of what the Babylonians were trying to do; and if so, which is appropriate?

4 The importance of number-writing

As we have already pointed out, Neugebauer and his generation were working on a restricted range of material. To some extent this was an advantage, in that it had some coherence; but even so, there were typical problems in determining provenance and date, because they were processing the badly stored finds of many earlier archaeologists who had taken no trouble to read what they had brought back. It is well worth reading the whole of Neugebauer's chapter on sources, which contains a long diatribe on the priorities and practices of museums, archaeological funds, and scholars:

> Only minute fractions of the holdings of collections are catalogued. And several of the few existing rudimentary catalogues are carefully secluded from any outside use. I would be surprised if a tenth of all tablets in museums have ever been identified in any kind of catalogue. The task of excavating the source material in museums is of much greater urgency[5] than the accumulation of new uncounted thousands of texts on top of the never investigated previous thousands. I have no official records of expenditures for expeditions at my disposal, but figures mentioned in the press show that a preliminary excavation in one season costs about as much as the salary of an Assyriologist for 12 to 15 years. And the result of every such dig is frequently more tablets than can be handled by one scholar in 15 years. (Neugebauer 1952, pp. 62–3)

5. Partly because, as Neugebauer has said earlier, tablets deteriorate when excavated and removed from the climate of Iraq.

There is probably better conservation of tablets now than when the above was written, but the long delay in publishing is still a problem[6]; and there are grounds for new pessimism now that one hears that tablets are being removed from sites in Iraq and traded, presumably with no 'provenance' or indication of place and date, over the Internet. (For a discussion by Eleanor Robson of these and other problems which face historians in the aftermath of the Iraq war see http://www.dcs.warwick.ac.uk/bshm/Iraq/iraq-war.htm.)

The best-known of the OB tablets can be seen as rather special. What can be recognized in them are several features that subsequent scholars felt could be identified as truly 'mathematical':

1. The use of a sophisticated system for writing numbers;
2. The ability to deal with quadratic (and sometimes, if rather by luck, higher order) equations;
3. The 'uselessness' of problems, even if they were framed in an apparently useful language, like the one above.

None of these characteristics are present (so far as we know) in the mathematics of the immediately preceding period, which in itself is noteworthy. Let us consider them in more detail.

The number system

You will find this described, usually with admiration, in numerous textbooks. The essence was as follows. Today we write our numbers in a 'place-value' system, derived from India, using the symbols $0, 1, \ldots, 9$; so that the figure '3' appearing in a number means 3, 30, 300, etc. (i.e. $3 \times 10^0, 3 \times 10^1, 3 \times 10^2, \ldots$) depending on where it is placed. The Babylonians used a similar system, but the base was 60 instead of 10 ('sexagesimal' not 'decimal'), and they therefore based it on signs corresponding to the numbers $1, \ldots, 59$—without a 'zero' sign. The signs were made by combining symbols for 'ten' and 'one'—a relic of an earlier mixed system, but obviously practical, in that what was needed was some easily comprehensible system of 59 signs. (see Fig. 4) You might, as an exercise, think of how to design one. The place-value system was constructed, like ours, by setting these basic signs side by side; we usually transliterate them and add commas, so that they can be read as in Fig. 5. '1, 40' means, then, what we would call $1 \times 60 + 40 = 100$; '2, 30, 30' means $2 \times 60^2 + 30 \times 60 + 30 = 7200 + 1800 + 30 = 9030$. 60 plays the role which 10 plays in our system.

There are, though, important differences from our practice. First, it is not explicitly clear that '30' on its own, with no further numbers involved necessarily means what we should call 30. It may mean $30 \times 60 (= 1800)$ or $30 \times 60^2 (=108{,}000), \ldots$. In a problem, it will be 30 somethings—a measurement of some kind, which is stated explicitly, for example, length or area in appropriate units; and this will usually make clear which meaning it should have. This is not the case with 'table texts' (e.g. the '40 times table'), which often concern simple numbers. Furthermore—compare our decimals—'30' can also mean $30 \times \frac{1}{60} = \frac{1}{2}$, and often does.[7] Or $30 \times \frac{1}{60} \times \frac{1}{60}$ and so on. If the answer was written as 30, you should—and this is an idea which we can recognize from our own practice—be able to deduce what '30' meant from the context.

6. Robson (1999) cites an example of a collection of OB proverb texts which were published in the 1960s with no acknowledgement by the scholarly editor that they had calculations on the back.

7. Although there were also symbols for the commonest fractions like $\frac{1}{2}$—see the above example—and (it seems) rules about when you used them.

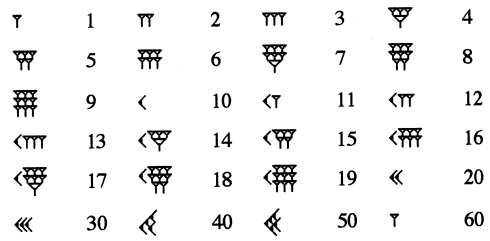


Fig. 4 The basic cuneiform numbers from 1 to 60.

Cuneiform	Transliteration	Decimal value
	1,15	75
	1,40	100
	16,43	1003
	44,26,40	160000
	1,24,51,10	305470

Fig. 5 How larger cuneiform numbers are formed.

You can find the details of how the system works in various textbooks; in particular, there are plenty of examples in Fauvel and Gray. (Notice that the sum which I gave above was one in which it was not needed—why?) Again following a general convention, modern editors make things easier for readers by inserting a semi-colon where they deduce the 'decimal point' must have come, and inserting zeros as in '30, 0' or '0; 30'. So '1, 20' means 80, but '1; 20' means $1 + \frac{20}{60} = 1\frac{1}{3}$. There would be no distinction in a Babylonian text; both would appear as '1 20'.

To help themselves, the Babylonians, as we do, needed to learn their tables. They were, it would seem, in a worse situation than us, since there were in principle 59 tables to learn, but they probably used short cuts. A scribe 'on site' would quite possibly have carried tablets with the important multiplication tables on them, as an engineer or accountant today will carry a pocket calculator or palmtop; and in particular the vital table of 'reciprocals'. This lists, for 'nice' numbers x, the value of the reciprocal $\frac{1}{x}$, and starts:

2	30
3	20
4	15
5	12
6	10
7, 30	8
8	7, 30
9	6, 40

Using this table it is possible to divide simply by multiplying by the reciprocal; dividing by 4 is multiplying by 15 (and of course thinking about what the answer means in practical terms—what size of number one should expect).

This way of writing numbers is so advanced and sophisticated that it has impressed most commentators, particularly mathematicians. The absence of a decimal point, as I have said, is not a serious problem in practical calculations; but it could raise questions when one is asked, for example, to take the square root (we will see this was done too) of 15. If '15' means $\frac{1}{4}$, then it has square root $30 = \frac{1}{2}$, but if it means '15', of course, it does not have an exact square root. However, the scribe would find the square root by looking in a table, and only one answer would appear, for any number.

The more serious problem which is often pointed out is the absence of a sign for 'zero'. In principle, $60\frac{1}{2}$, which should in our terms be '1 0 30' (one sixty, no units, 30 sixtieths) would be written '1 30', which could also mean '90' (or '$1\frac{1}{2} = 90 \times \frac{1}{60}$'). It is hard to know how often this caused confusion. One case is given by Damerow and Englund (in Nissen et al. 1993, pp. 149–50) of a scribe who is finding the powers of '1, 40', or what we would call 100. At the sixth stage one of the figures should be a '0', and is omitted. Hence this calculation, and the subsequent ones (he continues to 100^{10}) are wrong. However, you can see (why?) that this mistake would occur less often than in our decimal system if we happened to 'forget' zeros, and so confused 105 and 15.

Exercise 3. *Explain (a) how the table of reciprocals works, (b) why it does not contain '7'.*

Exercise 4. *Work out (1, 40)/(8) using the table, given that the reciprocal of 8 is 7, 30. (Check that this is indeed the reciprocal; and verify that you have the right answer, given that* $1, 40 = 100$ *in our terms.)*

Exercise 5. *(a) What is the square root of 15 if '15' means* 15×60*? (b) Show that, in Babylonian terms, there cannot be two different interpretations of a number which have different (exact) square roots.*

5 Abstraction and uselessness

The discovery of the sexagesimal system is sometimes described, by those who like the word, as a revolution. How it came about is unclear, but it does seem to have arisen quite suddenly out of a number of near- or pseudo-sexagesimal systems, around the beginning of the OB period. Damerow and Englund (Nissen et al. 1993, pp. 149–50) seem to consider it impractical, and claim it did not outlast the OB period—which is difficult to reconcile with their admission that it was used by the Greek astronomers. Here, indeed, we find our first example of the problem of connecting similar practices across time. Sexagesimals were used in Babylon in 1800 BCE, and again, mainly in astronomy, 1500 years later. (They were still being used—with multiplication tables—by Islamic writers in the fifteenth century CE (see Chapter 5) under the name 'astronomers' numbers'.) It seems almost certain that this was a direct line of descent from Babylon to Greece. More dubious claims are often made, though, in situations where the same result (e.g. 'Pythagoras' theorem') is known to two different societies—that there must have been either communication or a common ancestor. Such arguments are central (for example) to van der Waerden's fascinating but eccentric (1983); always controversial, they have to be evaluated on the basis of the evidence.

Equations

Here, if anywhere, the mathematicians can be allowed to judge what it is to be sophisticated. In examples like the one above, we see probably for the first time the idea of an unknown quantity—an

unweighed stone, in this case. The Egyptians were using the same idea a little afterwards, and may have arrived at it independently; but they did not succeed in the next step, which was a general method for solving quadratic-type problems. It makes sense to use this term, rather than 'quadratic equations', since the problems are very varied in nature; the 'quadratic equation' as we know it, a combination of squares, things, and constants, begins its history properly in the Islamic period. Fauvel and Gray's 1.E.(f) problem 7 starts:

I have added up seven times the side of my square and eleven times the area: 6; 15

In other words, we have a square, and we are told that seven times the unknown side x ($7x$) added to eleven times the area ($11x^2$) gives 6; 15 or $6\frac{1}{4}$. This leads to a simple quadratic equation, which we would write $7x + 11x^2 = 6\frac{1}{4}$, with answer $x = 0; 30 = \frac{1}{2}$. For how it is solved, which in particular shows where square roots were used, see Appendix A.

In addition to the relatively common equation texts, we have some texts which seem to show extra mathematical sophistication, some of which is still subject to debate. One is the notorious 'Plimpton 322'; for the original decoding of this see Fauvel and Gray and for a recent counter-argument, Robson (2001); we shall not consider this here, although it is an interesting introduction to the disagreements of historians. A simpler case is the 'square root of 2' tablet, which seems straightforward in its interpretation (Fig. 6). The picture shows a square; its side is marked 30 (or $\frac{1}{2}$), and the diagonal has two sexagesimal numbers marked. One is a good approximation to $\sqrt{2}$ (1, 24, 51, 10), the other to the diagonal $\sqrt{2}/2$ (42, 25, 35). Nearly the same sexagesimal numbers will appear again when we deal with Islamic mathematicians over 3000 years later; for now it is worth raising the question of what these numbers were used for, and how they were arrived at. In the absence of any written procedures, we can at least admire the result.

'Uselessness'

Sometimes mathematicians need to be reminded that mathematics, to be worthwhile, does not *have* to be useless; and they have often had a two-faced attitude on the subject, pointing (e.g. when

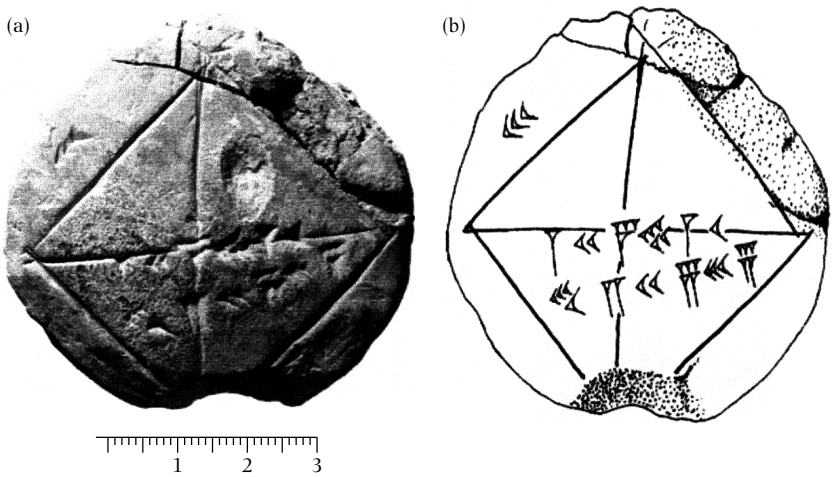


Fig. 6 The 'square root of 2' tablet YBC7289.

requesting a research grant) to results which were thought useless at the time and afterwards discovered to have an application. Well-known examples include Riemannian geometry and relativity, finite fields and the manufacture of CDs, etc. It has been a part of the case for the seriousness of Babylonian mathematics that their problems, while apparently practical, were clearly not designed for the real world. Rather, they were exercises in technique dressed up in practical language (because that was the only language available). The point is often made, and can hardly be contested. Our first example (stone-weighing) is a good illustration. So is the quadratic equation above—it would be hard to think of circumstances in which one would want to add lengths to areas, and the Greeks, with a more strict idea of geometry, did not have a language in which to do it. In another example often cited, the student is given the amount of earth required to fill a ramp, and asked to find its dimensions—exactly the opposite of the practical question. No Babylonian text theorizes this impracticality as such, or makes a virtue of it; while Plato, as we shall see, makes a distinction between real mathematics and that which is used by artisans, the Babylonian scribes to all appearances were trained for a career of useful tasks by solving problems with no application.

What was the point of this? To answer this question would require some thought about what the 'point' of any mathematical procedure is. At one level, we can imagine that the ability to deal with increasingly difficult problems, regardless of their meaning, could be used as an examination-type filtering mechanism within the scribal schools, marking off the bright students from the mediocre ones; or, outside the schools, it could be a form of competition between 'freelance' scribes (they existed too) who were trying to attract clients. This virtuosity is part of a whole package of skills which were important for self-definition and for status:

> According to the 'Examination Text A', the accomplished scribe must know everything about bilingual [that is, Sumerian/Akkadian] texts; he must know occult writings, and occult meanings of signs in Akkadian as well as Sumerian; he must be familiar with the concepts of musical practice, and he must understand the distorted idiom of various crafts and trades. Into the bargain then comes mathematics . . . All that, as a totality, has a name (of course Sumerian): nam-lú-ulù, 'humanity'. (Høyrup 1994, p. 65)

This 'external' explanation does not, however, account for the particular choice of impractical quadratic equations for the display of accomplishment. Here we have, almost, an example of Kuhn's 'normal science'. A technique—the solution of linear and quadratic problems using sexagesimal numbers and tables—becomes available, for reasons which are unclear; and the scholars who make up the community are defined by their ability to solve puzzles using the technique. In addition, they may find the problems interesting or challenging, in a career dedicated to routine tasks (but here we are indeed speculating). In principle, hard puzzles can generate harder ones without limit; in terms of the historical record, it seems that either invasion or loss of interest or both put an end to the practice.

The idea of 'uselessness' is one which needs to be treated with some care, however. It is easy, considering some of the OB calculations, to deduce that their apparent practicality is a fake and that they are simply occasions for what Høyrup calls displays of 'scribal virtuosity'. This is the traditional view, and many of the texts support it. However, Robson's detailed new publication (1999), containing a wide variety of tablets, is the basis for arguing a more complex view. An example is a long tablet (BM96957 + VAT6598) containing a succession of problems about brick walls. These depend crucially for their solution on one of the basic scribes' numbers: the conversion factors from (volume of wall) to (number of bricks in the wall) and back—an eminently practical figure, and one which was certainly often used. The problems start with questions which give the

measurements of the wall (length, width, and height), and ask how many bricks. Naturally, the OB scribes (like us) used different units for width (cubits, compare inches), and length and height (nindan, compare feet); the calculation was therefore not always a straightforward one leading to a certain number of cubic nindan and dividing by the number of bricks in a cubic nindan. Such a question seems both simple and practical, and just the kind of thing which a scribe in the brick-wall construction trade might be asked. However, question 5 on the same tablet is:

> A wall. The height is $1\frac{1}{2}$ nindan, the bricks 45 sar_b [brick measure]. The length exceeds the width of the wall by 2; 20 nindan. What are the length and the width of my wall? (Robson 1999, p. 232)

The details of brick-measure and height belong to everyday practice, but it seems very unlikely that one would ever need to answer a question of this type in a practical situation. Somewhere along the list of problems on the tablet a link to real-world wall-building has been broken.

Exercise 6. *If you are told that 72 sar_b of bricks occupy a volume of 1 cubic nindan, (a) show that this is equivalent to a quadratic problem and (b) find the answer.*

6 What went before

The last example shows that there may still be more to learn about the OB period. In recent times, a much fuller picture has emerged of the earlier period of mathematics, and it is currently perhaps the most interesting area of research. What we have is still more a series of snapshots than a record of discovery; in archaeology it is almost unknown to find an innovation which can be accounted for, much less attributed to an 'author'; but it allows us to question the idea of Babylonian mathematics as the earliest serious practice, based on the criteria I have given.

In the first place, we know that the profession of scribe, and the scribal schools, existed for some time before (the usual estimate is around 2500 BCE for the beginning of the institution). Even in this very early period, when the number system, while quite clear and flexible, was much less advanced than the sexagesimal one, we find that the schools had discovered the idea of setting problems which were both difficult and useless, if in a different way—in fact, the mixed nature of the number system made questions which we might think easy harder. They require simple division of a very large (i.e. impractical) number by a number which makes problems. Specifically,

> that the content of a silo containing 2400 'great gur', each of 480 sila, be distributed in rations of 7 sila per man (the correct result is found in no. 50: 164,571 men, and a remainder of 3 sila) . . . (Høyrup 1994, p. 76)

A sila being roughly (it is thought) a litre, we are dealing with over a million litres, and the proposed division by 7 (with remainder!) is an exercise in obsessive accuracy rather than a practical problem. In the words of Jöran Friberg

> the obvious implication is that the 'current fashion' among mathematicians about four and a half millennia ago was to study non-trivial division problems involving large (decimal or sexagesimal) numbers and 'non-regular' divisors such as 7 and 33. (Cited in Høyrup 1994, p. 76)

Friberg uses the term 'mathematicians' to describe those scribes and teachers who discussed such problems; and such a usage not only sets the origin of mathematics as an independent practice much earlier, but makes it appear much more 'trivial' to us. If the Babylonians can be grudgingly

admired for solving quadratic equations, can we extend a similar recognition to the scribes of Fara for doing rather long divisions? There has indeed been quite a controversy about what the Fara scribes were supposed to do in answering the question; see Powell in Fauvel and Gray 1.E.5, or for a more recent view, Melville, 'Ration Computations at Fara: Multiplication or Repeated Addition' in Steele and Imhausen (2002). Again, this question is perhaps best left unanswered, or as a point for discussion. Friberg would probably justify calling the scribes 'mathematicians' not in terms of their use of unrealistic examples, but in the formation of a community—again that Kuhnian word—with a common project, whose language was a language of numbers. Training for practical purposes seems, here too, to have generated a class of impractical exercises, if entirely different from those which followed 500 years later.

However, this impracticality, characteristic of the school-texts which have survived, disappears when we look at a different family of texts, the accounts from the harsh period known as Ur III, which were the work of *practising* scribes and administrators. (What kind of texts survive from which period is at least partly chance, depending on the kind of site excavated.) Dating from the twenty-first century BCE, these are in time between the Fara texts and the OB ones, and they are both utilitarian and highly 'mathematized'. The period, under King Šulgi, was one of increasingly rigid centralized control of production; the aim, for a variety of industries—seed production, cattle raising, fishing, milling, and so on—is to calculate the expected yield and the extent to which the farmers or managers fulfil their targets. Analogies with old Soviet planning or indeed modern Western management come to mind. Accounts were complicated by the fact that almost any quantity had a special system of units to measure it. However, the scribe is, on the whole, up to the calculation; as usual, there are tables of conversion factors to help. Here is an example which, according to Damerow and Englund (Nissen et al. 1993, pp. 141–2), represents 'the calculation of the harvest yield of the province of Lagash for the third year recorded in the text' (Fig. 7). We begin

Fig. 7 The tablet recording harvests from Lagash, AO3448.

by setting out the area:

1 (šár-gal) 1 (šar'u) 1 (šár) 1 (bùr) field surface

Then follow the 'targets'; the amount which this area should produce:

the barley involved: 3 (šar'u) 5 (šár) 3 (geš'u) 3 (u) gur

Finally, the actual amount produced, and the shortfall:

Therefrom
2 (šar'u) 1 (šár) 4 (geš'u) 7 (géš) 4 (u) 2 (gur) 1 (barig) 4 (bán) gur delivered.
Deficit: 1 (šar'u) 3 (šár) 4 (geš'u) 3 (géš) 2 (u) 7 (gur) 3 (barig) 2 (bán) gur

A first observation is that a quite unnecessary number of units of measurement seem to be involved (and there are yet more . . .). They are of course exotic to us, but at 4000 years' distance we can expect that. The first row gives the area of the fields producing barley. According to Nissen et al. 1993, pp. 141–2, 1 bùr is about 6.3 hectares; and

1 šár = 60 bùr
1 šár'u = 10 šár
1 šár-gal = 6 šár'u.

The total area is therefore (work it out) 4261 bùr or 26,844 hectares. The calculation of 'the barley involved' in the second row is the 'target'; it assumes that an area of 1 bùr produces 30 gur (9000 litres) of grain. For the grain measure we have:

(1 bán = 10 litres)
1 barig = 6 bán
1 gur = 5 barig
1 u = 10 gur
1 géš = 6 u
1 geš'u = 10 géš
1 šár = 6 geš'u
1 šar'u = 10 šár

As you can see, the units do not proceed by uniform steps, and even multiplying the area by the factor of 30 gur and translating it into volume units to get the target volume is quite complicated. Hence the figures 1, 1, 1, 1 in the first row translate into 3, 5, 3, 3 in the second.

We now have to subtract the actual output from the target; and the actual figure involves a rather excessive eight units of measurement (all the ones listed above).[8]

This should be enough to convince you that, while Ur III accountants' arithmetic was 'elementary', it was far from simple, and considerable skill was required to get the deficit right. (Happily, there was, it seems, not always a deficit; apparently in the first of the three years listed on the tablet the harvest was more than expected. On the other hand—see Englund (1991)—the targets set for labourers in factories seem generally to have been unrealistically high and calculated

8. But before we condemn the Sumerians for their complexity, it should be noted that schools in England 50 years ago taught a system of 8 units of length—line, inch, foot, yard, rod (or pole, or perch), chain, furlong, mile—and that the factors relating them were more complicated than the Sumerian 5s, 6s, and 10s.

to ruin their overseers to the greater profit of the state.) Of course, we still sometimes face problems of this multi-unit kind, such as when we try to find the time lapse between 1.25 p.m. on January 28 and 11.15 a.m. on February 2 in days, hours, and minutes; but these are rare and the metric system is reducing them.

Exercise 7. *Trace the calculation of 'the barley involved' through and check it.*

Exercise 8. *Calculate the deficit, using the table of barley measures, and find the two places where the scribe has made a mistake.*

7 Some conclusions

The above example is worth some consideration, if only because you will not often find such work discussed. In a sense the mathematics is trivial, in another clearly not; it is highly organized, and it needs to be accurate (although mistakes were not uncommon). It is as much a product of the bureaucracy and the organization of scribes as are the more interesting and mathematically impressive OB examples with which we started, and which you will usually meet; and its basic tools—multiplication and subtraction, with 'conversion factors' to make it more difficult—have also had a long history, and are still with us. The rationality of the OB system is often mentioned to boost its credentials as the earliest real mathematics, as is the fact that it survives in our measurements of time (minutes and seconds) and angle (degrees, minutes, and seconds). However, even today we often in practice find we have to operate with mixed systems of measurement, and work out the relevant sums as best we can. We could call such a procedure irrational (but on what grounds?); it does not make the mathematics easier. Only those who have never made mistakes in such conversions (e.g. miles and yards to and from metric) can dismiss them as not mathematical.

Appendix A. Solution of the quadratic problem

The solution given (from Neugebauer, also in Fauvel and Gray I.E.(f), problem 7) is as follows. The intrusive semicolons have been omitted; you will have to work out where they should come. On the other hand, the procedure is translated into algebraic notation in brackets, so that it can be followed more easily.

You write down 7 and 11. You multiply 6,15 by 11: 1,8,45. (Multiply the constant term by the coefficient of x^2.)

You break off half of 7. You multiply 3,30 and 3,30. (Square half the x-coefficient.)

You add 12,15 to 1,8,45. Result 1,21. (12,15 is the result of the squaring, so the 1, 21 is what we would call $(b/2)^2 + ac$, if the equation is $ax^2 + bx = c$.)

This is the square of 9. You subtract 3,30, which you multiplied, from 9. Result 5,30. (This is $-(b/2) + \sqrt{(b/2)^2 + ac}$; in the usual formula, we now have to divide this by $a = 11$, which we proceed to do.)

The reciprocal of 11 cannot be found. By what must I multiply 11 to obtain 5,30? The side of the square is 30. ('Simple' division was multiplying by the reciprocal, for example, dividing by 4 is multiplying by 15, as we have seen. If there is no reciprocal, you have to work it out by intelligence or guesswork, as is being done here.)

Solutions to exercises

1. I shall not answer, while exercise 2 is answered in the text.

3. If the number in the left column is x, that in the right column is y where $x \,.\, y = 1$. How does this work? For example, $4 \,.\, 15 = 60$ (which is 1, or 1, 0 if you want to use the notation of modern translation), and $8 \times (7, 30) = 8 \times 7 + 8 \times (0, 30) = 56 + 4 = 60$ again. More generally, one could think of x and y as solving some equation $x \,.\, y = 60^k$; the value of k is immaterial, since in Babylonian notation we cannot, for example, tell the different answers 15 and '0,15' ($= \frac{1}{4}$) apart.

This process works if such a y can be found, that is, if x divides some power of 60 exactly. (More exactly, we choose an interpretation of x which is a whole number, not a fraction.) This will be true if (and only if) all the factors of x are 2s, 3s, and 5s. It will therefore not work for 7.

4. $(1, 40)/(8)$ would be calculated as $1, 40 \times 7, 30$ (times the reciprocal). Use the formulae: $7 \times 40\,(= 280) = 4, 40$ and $30 \times 40 = 20, 0$; and take care of place value. You find the product is

$$7, 0, 0 + 4, 40, 0 + 30, 0 + 20, 0 = 12, 30, 0$$

If you were a Babylonian scribe, and knew that the '$1, 40$' meant 100, you would have no difficulty in interpreting this answer as $12\frac{1}{2}$.

5. 1. Of course, $15 \times 60 = 900$, which is a square, indeed the square of 30. So the statement 'square root of 15 is 30' is true also for this interpretation of 15.

2. This is a standard fact about place-value systems (unless the 'base' is a square). The different interpretations of any number are, say a basic 'x' and $x \times 60^k$. If $x = y^2$, then (since 60 is not a square), $x \times 60^k$ is a square if and only if k is even, say $k = 2l$; when $x \times 60^k$ is the square of $y \times 60^l$. So, in Babylonian terms, the square root of x is always y.

6. The Babylonian answer is given in Robson (2000, p. 232); it is hard to follow, since the text switches between 'sar_v' (a volume unit), nindan, and cubits (at 12 cubits to a nindan). Above, we have given the conversion factor from bricks to volume in cubic nindan instead of sar_v to reduce the number of measures. Here is a simplified version of the answer in our notation.

45 sar_b of bricks occupy $\frac{15}{24}$ cubic nindan, so that is the volume. The height is $1\frac{1}{2}$, so the area (length times width) is $\frac{5}{12}$ square nindan. If l is the length and w the width, $l = w + \frac{7}{3}$, and $lw = \frac{5}{12}$; so

$$w\left(w + \frac{7}{3}\right) = \frac{5}{12}; \qquad 12w^2 + 28w = 5$$

Clearly this is quadratic, and the solutions are $w = \frac{1}{6}$ and $w = -\frac{5}{2}$. Realistically, the wall has width $\frac{1}{6}$ nindan (2 cubits) and length $2\frac{1}{2}$ nindan.

7. We could, as above, reduce everything to the simplest units; but that is probably not what was done. To proceed 'properly', start from the right (this may not have been usual, but it is our habit). 1 bùr gives 30 gur or 3 u, from the table. 1 šár (60 bùr) therefore gives 60×3 u, or 3 geš'u (since there are 60 u in 1 geš'u). 1 šar'u gives 10 times this, which is 30 geš'u, or 5 šár. And finally, 1 šár-gal gives six times this, which is 30 šár, or 3 šar'u.

8. We now have to subtract. This time again, it is more correct to borrow along the line. However, since you can risk making mistakes quite easily given the number of 0s in the top row, I will reduce everything to bán. The amount due is 3,834,900 bán, the amount delivered is 2,353,870 bán; and the deficit is 1,481,030 bán; which is 1,080,000 (1 šar'u) + $3,240,000 = 3 \times 108,000$ (3 šár) + $72,000 = 4 \times 18,000$ (4 geš'u) + $3600 = 2 \times 1800$ (2 géš) + $1200 = 4 \times 300$ (4 u) + $210 = 7 \times 30$ (7 gur) + $18 = 3 \times 6$ (3 barig) + 2 bán. This agrees with the scribe's calculation apart from the figures for u and géš.